Localized finite-amplitude disturbances and selection of solitary waves

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It turns out that evolution of localized finite-amplitude disturbances in perturbed KdV equation is qualitatively different compared with conventional small-amplitude initial conditions. Namely, relatively fast solitary waves, with one and the same amplitude and velocity, are formed ahead of conventional chaotic-like irregular structures. The amplitude and velocity of the waves, obtained from the asymptotic theory, are in excellent agreement with numerics.

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I. INTRODUCTION

The subject of our study is the following model equation:

$$\eta_t + 2a\,\eta\,\eta_x + b\,\eta_{xx} + c\,\eta_{xxx} + d\,\eta_{xxxx} + \sigma(\eta\,\eta_x)_x = 0, \tag{1}$$

where a,b,c,d,σ are constants, and $0 \leqslant x \leqslant L$. This model appeared in context of thin liquid films [1,2], Benard-Marangoni convection in shallow layers [3,4], and other systems [5]. The similar equation governs evolution of longitudinal strain waves in an elastic rod embedded in viscoelastic external medium [6]. Typically, $\eta(x,t)$ denotes surface deviation from its steady-state flat position. The coefficients in Eq. (1) depend upon parameters of the system. Solutions of Eq. (1) allow us to shed a light into subtle problems of applicability of long-wavelength weakly nonlinear asymptotic models to the original physical problems [5].

One may consider Eq. (1) as a slightly dissipation-perturbed KdV equation [4,7,8], where b,d,σ are much smaller than a and c. As a result, Eq. (1) possesses the exact solutions in the form of localized nonlinear waves of permanent form, or dissipative solitary waves; the existence of an exact solution requires constraints on the coefficients, such as $c = 2\,a\,d/\sigma$ [5,9,10]. Solutions of Eq. (1) in form of asymptotic travelling solitary wave [4,7,8] are stable against small arbitrary perturbations [7]. More complicated travelling wave solutions were studied using the phase diagram analysis [11,12]. Ultimately, the mentioned works deal with the ODE reduction of Eq. (1) through travelling wave assumption. As far as we know, the unsteady process of solitary wave formation was not studied.

Equation (1) appears also as an extension of the Kuramoto-Sivashinsky (KS) model, with small dispersion and backward quadratic diffusion, i.e., for small c and σ [5]. As was found in Ref. [5], for sufficiently small damping ($\sigma < \sigma_c$), any small-amplitude initial noiselike function will explode in a finite time. For the larger damping, $\sigma > \sigma_c$, small-amplitude initial data evolves into finite-amplitude irregular patterns. As a result, critical value of damping σ_c divides the region of unconditional blow-up, from the region where small-amplitude disturbances give rise to finite-amplitude patterns. For higher damping, patterns become spatially ordered. Finally, for the damping large enough, all modes are linearly stable. However, even the stable patterns may be destroyed if strongly perturbed [5]. For the relatively weak dispersion ($|\beta| \leqslant 0.1$, Fig. 1 in [5]), it was concluded that the solitary waves are in the blow-up parameter range, and cannot materialize [5].

The subject of the present work is to find how these solitary waves *can* materialize, and to analyze the process of their formation and selection. It turns out that for sufficiently strong dispersion the solitary waves found in [5,9,10] are realizable. We develop the theory to describe the unsteady process of solitary waves selection in the case when b,d,σ are small. These solitary waves can be formed only from the localized, finite-amplitude initial conditions. The general mechanism of the waves survival is similar to that found in Ref. [13]: the solitary waves travel faster, and therefore escape the destructing influence of growing disturbances behind. As a result, depending on initial conditions, irregular and solitary waves may coexist simultaneously, and be separated in space.

II. ANALYTICAL DESCRIPTION OF DISSIPATIVE SOLITARY WAVES FORMATION

The key point of our study is to consider model (1) as KdV-perturbed equation. We are interested in extraction of the leading-order terms describing the slow variation of the soliton's parameters. Assume that $(b,d,\sigma) = \varepsilon(B,D,\Sigma)$, and $\varepsilon \ll 1$. We follow the uniformly valid asymptotic procedure developed in Ref. [14]. We suggest that η depends upon a fast variable ξ and a slow time T, such as

$$\xi_x = 1, \quad \xi_t = -V(T), \quad T = \varepsilon\, t.$$

Then Eq. (1) becomes

$$c\,\eta_{\xi\xi\xi} - V\,\eta_\xi + 2a\,\eta\,\eta_\xi + \varepsilon\,[\,\eta_T + B\,\eta_{\xi\xi} + D\,\eta_{\xi\xi\xi\xi} + \Sigma(\eta\,\eta_\xi)_\xi] = 0. \tag{2}$$

The solution of Eq. (2) is sought in the form

$$\eta(\xi,T) = \eta_0(\xi,T) + \varepsilon\,\eta_1(\xi,T) + \cdots. \tag{3}$$

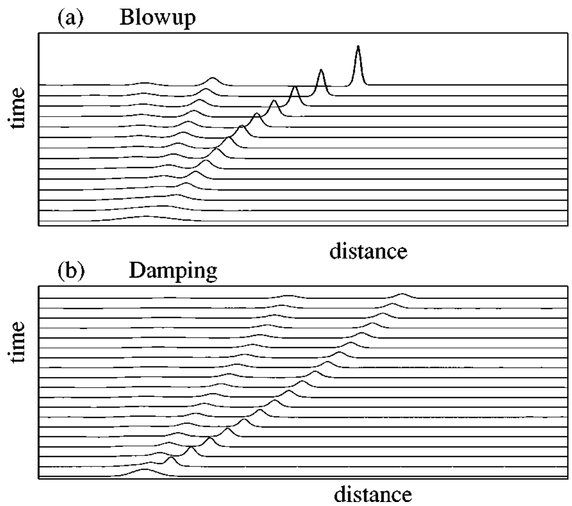

FIG. 1. Blow-up (a) and damping (b) of the initial conditions.

We are interested in studying localized solutions vanishing together with its derivatives at $|\xi|\to\infty$. Then the leading order solution is

$$\eta_0=\frac{6\,c}{a}S^2\cosh^{-2}(S\,\xi),\qquad S=S(T). \tag{4}$$

Here $S(T)$ is a slowly varying function, and $V=4c\,S^2$. From the solvability condition for nonsingular solution of Eq. (2) at order ε [14] follows:

$$S_T'=\frac{8}{15}S^3(B+P\,S^2). \tag{5}$$

Here

$$P=\frac{4}{7}\left(6\frac{c}{a}\Sigma-5D\right). \tag{6}$$

We omit standard arguments of the asymptotic procedure, giving only its short sketch. Function $\eta_1(\xi,T)$ is a solution of linear inhomogeneous ODE; it will contain therefore free parameters depending on T such as $S(T)$ in the leading order problem. Its definition allows us to satisfy solvability condition in the next order problem and to avoid secular terms in the asymptotic expansion. Higher order approximations may be studied similarly.

Equation (6) allows us to find the variation of the amplitude and velocity of the soliton in time. The behavior of S depends on the signs of B and P and on the value of $S_0\equiv S(T=0)$. Indeed, when both B and P are positive S diverges while for both negative values it will vanish. For $B<0$, $P>0$ the parameter S vanishes if $S_0<\sqrt{-B/P}$ while it diverges if $S_0>\sqrt{-B/P}$.

In the case of divergence, the explicit expression of the blow-up time (''time-of-life'') is

$$T=\frac{4\,P}{15B^2}\ln\left|\frac{S_0^2P}{B+P\,S_0^2}\right|+\frac{4}{15B\,S_0^2}. \tag{7}$$

Note that in this case the initial assumption about slow evolution of soliton's parameters is violated: near blow-up, soliton changes fastly. Nevertheless, it is interesting to compare this formal prediction with numerically evaluated dynamics, which is done in the next section.

In the case of damping, S vanishes when T tends to infinity. The most interesting case occurs when $B>0$, $P<0$, when S tends to $\sqrt{-B/P}$ independent of S_0. Relation (5) may be directly integrated giving the implicit dependence of S on T:

$$T=\frac{4\,P}{15B^2}\ln\left|\frac{S_0^2(B+P\,S^2)}{S^2(B+P\,S_0^2)}\right|-\frac{4(S_0^2-S^2)}{15B\,S^2\,S_0^2}. \tag{8}$$

Note that expression (8) provides an analytical description of the time-dependent process of the selection of the solitary wave (4). As was mentioned in Ref. [5], for sufficiently large disturbances the backward diffusion term $(\eta\eta_x)_x$ becomes dominant, and leads to blow-up. Equation (5) does not allow us to capture this subtle effect; probably, the asymptotic procedure should be carried further to describe the phenomenon.

In the case of selection, if we additionally assume $c=2aD/\Sigma$, the asymptotic dissipative soliton (4) will tend to the exact travelling solitary wave solution of Eq. (1) [9,10]. Noteworthy is that this cannot be attained from infinitesimal initial disturbances [5]. The result of our time-dependent analysis agrees with the results of earlier stability analyses of quasistationary solitary waves [4,7].

In the next section, we analyze Eq. (1) numerically to compare with analytical results given in this section. We are especially interested in the process of selection of solitary waves.

III. NUMERICAL STUDY OF SOLITARY WAVES

To simulate Eq. (1), the following numerical technique has been employed. We define the typical wave number $k_c=\sqrt{b/2d}$ corresponding to the most unstable linear mode in Eq. (1). The corresponding wavelength is $\lambda_c=2\pi/k_c$. The length of spatial domain was chosen to be $256\lambda_c$, i.e., rather long. At the same time, the number of discretization points was chosen to be 4096, i.e., λ_c is covered by 16 points. The latter ensures fair resolution of the whole solutions computed. Periodic boundary conditions have been used for simulations. The pseudospectral technique was employed for the spatial discretization and the Runge-Kutta fourth order scheme for the time advance. The time step was chosen to be 0.01. The tests with smaller time steps and better resolution gave indistinguishable results. The control of the simulations in the Fourier space shows the very good resolution of the computed solutions.

Note that the previous section describes the evolution of soliton's parameters; so, formally, we can apply the obtained results only to the initial data in the form of solitons. It is clear, however, that this type of initial conditions is very restrictive. We therefore use general localized initial conditions (usually, in the from of Gaussian). Since b,d,σ are small, at the first stage, the dynamics of the initial conditions is expected to be governed by the KdV equation and the solitons will be formed on this preliminary stage. The selection process, which is the subject of our interest, is expected to appear at later times, when solitons are already formed. Therefore, we can compare the proposed theory with numer-

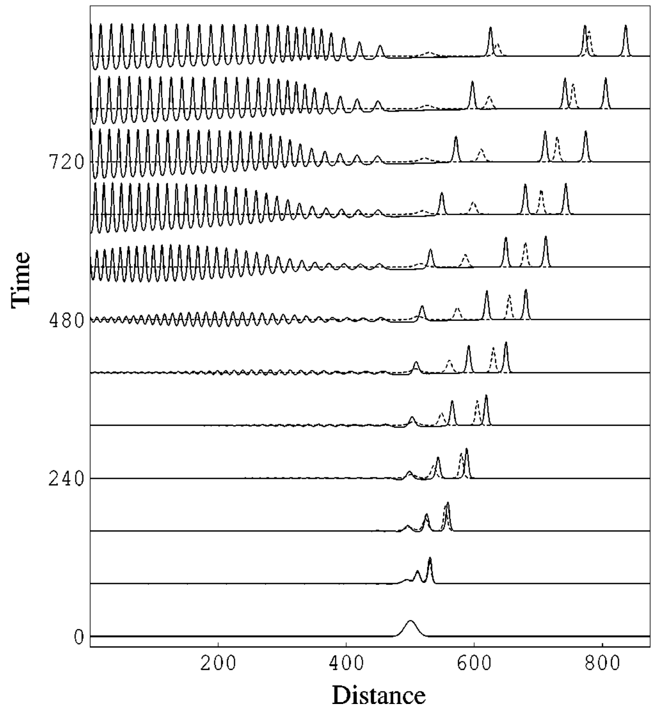

FIG. 2. Selection of solitary wave from ''below''; initial condition is a Gaussian profile with amplitude 0.3 and width 36 units; only a part of the long spatial domain is shown here.

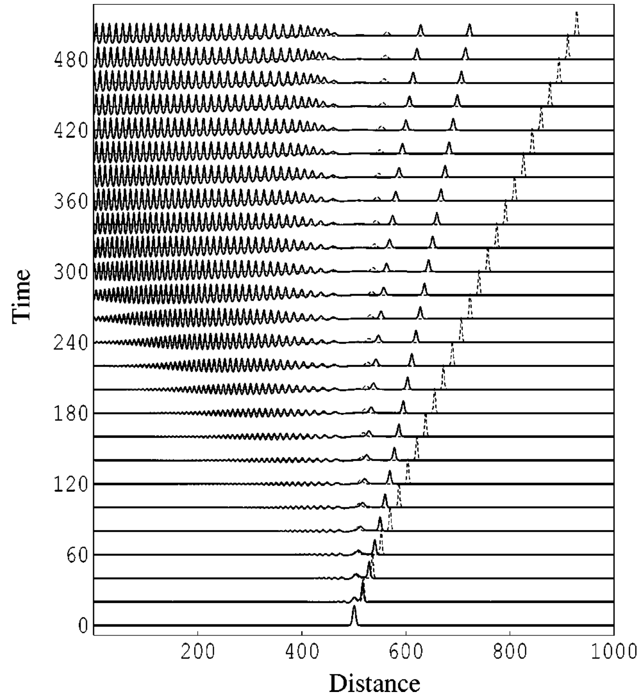

FIG. 3. Selection of solitary wave from ''above''; initial condition is a Gaussian profile with amplitude 1 and width 12 units; only a part ob the long spatial domain is shown here.

ics even for rather arbitrary localized initial conditions.

First, it has been checked that the damping, and the tendency to blow-up have been observed at the values of coefficients found in the previous section. Numerical results are shown on Fig. 1. For the damping, $\varepsilon=0.1$, $a=1$, $B=-1<0$, $c=1$, $D=6/5$, $\Sigma=-2$, so $P=-72/7<0$. In the case of blow-up, $\varepsilon=0.1$, $a=1$, $B=1>0$, $c=1$, $D=6/5$, $\Sigma=2$, so $P=24/7>0$. For the blow-up, we found that the pulse tends to grow rapidly at the time t^* rather close to the predicted $t^*=T^*/\varepsilon$ from Eq. (7) (details of numerics are omitted). As was mentioned, the description of the fast blow-up process formally contradicts the initial assumptions of slow variation of soliton's parameters. It is remarkable that the asymptotic procedure captures blow-up, and is valid much further than formal parameter range.

Selection of dissipative solitons occurring at $B>0, P<0$. We present here the results of simulations for the following parameter values: $\varepsilon=0.1$, $a=1$, $B=1>0$, $c=1$, $D=6/5$, $\Sigma=-2$; therefore, $P=-72/7$. The asymptotic value S_∞ of $S(T)$ at $T\to\infty$, obtained from Eqs. (4), (6), is $S_\infty=\sqrt{-B/P}=0.312$. The asymptotic amplitude of the solitary wave is $6cS_\infty^2/a=0.583$, and its asymptotic velocity is $V=4c\,S_\infty^2=0.389$. We consider both the selection occurring from ''below'' when the magnitude of an initial Gaussian pulse is smaller than that of the eventually selected solitons, Fig. 2, and the selection from ''above'' when the selected solitons amplitude is smaller than that of the initial pulse magnitude, Fig. 3. Note that only a part of long spatial domain is shown on Figs. 2 and 3 to make the evolving structures distinguishable. One can see in Fig. 2 that up to the time $t\sim 120$ an initial Gaussian pulse with the magnitude $0.3<0.583$ and width 36 breaks into a train of three localized pulses aligned in row of decreasing magnitude. Due to smallness of ε, the influence of the dissipative non-KdV terms is small at this stage. This may be seen by comparison of solutions of Eq. (1), shown as solid lines, with pure KdV case, $B=D=\Sigma=0$, shown by dashed lines. At later stages the initial pulse transforms into a train of the solitary waves. For nonzero ε, amplitude and velocity of each solitary wave tend to the values 0.585 and 0.38 in agreement with the theory of single solitary wave selection, while each of three KdV solitons continues propagation with its own amplitude and velocity. Unequal spacing between equally high crests reflects the original separation of the solitons in the KdV stage when higher solitons travel faster. The tail behind the train of solitary waves appear as a result of long wave instability [5,15]. These irregular waves cannot be described by the theory proposed in the previous section. The solitary waves have higher velocity than the velocity of growing wave packet. As a result, the solitons escape the destructive influence of the tail behind. This mechanism was reported in Ref. [13] in context of close dynamic equation. We see on the last stages that the magnitude of the tail saturates. All these structures are quite well resolved, robust and similar to those found in Refs. [5,15].

The selection process realized from ''above'' is shown in Fig. 3 when initial Gaussian pulse has magnitude $1>0.583$ and width 12. Two equal solitary waves with amplitude 0.585 and velocity 0.38 appear as a result of *decrease* of the magnitude of the initial pulse. Again the comparison with pure KdV case is shown by dashed lines. All features of the selection process are similar to the selection from ''below.''

Note that the localization of the disturbances in space has a crucial role here. If we consider conventional small-amplitude noise as initial condition, similar to those considered in Ref. [5], the growing long-wavelength disturbances will destroy the possible formation of the solitary waves (the results of numerical simulations are not shown here). Besides this, it is important to mention the significance of the periodic boundary conditions. After sufficiently long time, solitons, moving faster than the irregular waves, will reach the end of the computational domain, will appear at the beginning of the domain, will further reach and collide with the irregular waves, and will be ultimately destroyed (not shown here). This is the reason for consideration of long spatial interval — to allow the waves to evolve naturally and to separate, without collision. Simulations of Eq. (1) with other values of parameters such that $B>0, P<0$, which are not reported here, show the same features of the selection process as described above.

IV. CONCLUSION

We considered Eq. (1) in the case when it could be regarded as a perturbed KdV equation, i.e., when b,d,σ are small. Solutions in the form of soliton with slowly varying parameters are considered. Solvability condition, based on standard asymptotic technique, leads to explicit equation describing the evolution of soliton's parameters. It turns out that three scenario are possible: blow-up, damping, and selection. In the case of blow-up, the asymptotic technique predicts the ''time-of-life'' which is quite close to numerically evaluated one. We conclude therefore that in this case the asymptotic technique is valid out of the range of its formal applicability. In the case of selection, the theory predicts the amplitude and velocity of the selected solitons with excellent accuracy. It is interesting that in this case the selection of the solitary waves is accompanied by development of irregular waves behind the solitary waves. The solitons travel faster, and escape therefore the destruction by the tail [13].

As was found in Ref. [5], for small dispersion (small c) the solitary waves fall in the range of blow-up, and therefore cannot materialize. We conclude that for sufficiently strong dispersion (when KdV-part is a leading player, as considered here), the solitary waves can materialize. Besides this, we revealed that solitary waves and irregular waves found in Refs. [5,15] may coexist simultaneously, being separated in space.

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